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On the running of the bare coupling in SU(N) lattice gauge theories

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Abstract: Interpreting the way that the SU(3) bare lattice coupling runs with the lattice spacing is complicated by the fact that there is a smooth cross-over region in which the strong coupling expansion transforms into a weak-coupling one. For $N \geq 5$, however, there is a first order bulk transition that cleanly separates the strong and weak coupling regimes. We find that in this case the calculated string tension can be readily fitted throughout the weak coupling region by a standard 3-loop expression modified by lattice spacing corrections of the expected form. While our fits demand the presence of the latter, they do not constrain the perturbative coupling scheme enough to enable us to extract a usefully accurate value of $a(\beta)$ in units of $\Lambda_{\overline{MS}}$. To resolve this ambiguity we turn to SU(3) where we use the Schrodinger Functional coupling scheme to extract a value of $r_0\Lambda_{\rm SF}$ as a benchmark. We then find that the Parisi mean-field improved coupling scheme closely reproduces this result. We also develop a comparison between different schemes that does not rely on the calculation of any physical quantity and which can therefore be applied much further into weak coupling. Again the Parisi scheme is favoured over the others that we compare. Using the mean-field scheme we have fitted the values of the string tension $a^2\sigma$ that have been calculated for $2 \leq N \leq 8$, to obtain $\Lambda_{\overline{MS}}/\sqrt{\sigma} = 0.503(2)(40) + 0.33(3)(3)/N^2$ for $N \geq 3$, where the first error is statistical and the second is our estimate of the systematic error from all sources.

Keywords: Asymptotic Freedom, 1/N Expansion, Lattice Gauge Field Theories, Renormalization Group.

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1. Introduction

Consider SU(N) gauge theories discretised onto a hypercubic lattice of spacing a, on a 4-torus, with SU(N) matrices, U_l, assigned to the links l, and with the standard Wilson plaquette action. The partition function is

$$Z = \int \prod_l dU_l \exp\left\{-\beta \sum_p \left\{1 - \frac{1}{N}\mathrm{ReTr}U_p\right\}\right\} \tag{1.1}$$

where U_p is the ordered product of the SU(N) matrices around the boundary of the plaquette p. (Although we shall be using the plaquette action in this paper, a parallel analysis could be carried out for any other lattice action.) The parameter β in the lattice partition function is proportional to the inverse lattice bare coupling. This defines a running coupling on the length scale a, in what is often called the lattice scheme:

$$\beta = \frac{2N}{g_L^2(a)}. \tag{1.2}$$

In practice one frequently wishes to compare physical quantities that have been calculated over a similar range of β, but not at precisely the same values. For example the Sommer parameter r_0 [1] or the deconfining temperature T_c [2] in terms of the confining string tension σ. One then needs to interpolate such quantities to common values of β. It is well-known that simple perturbative interpolations will not fit the calculated values. Thus one is typically reduced to using, for example, interpolating polynomials [3] that bear no relationship to the weak-coupling expansion, or, if one tries to use a power series

in $g_L^2 \propto 1/\beta$, one finds that the higher order fitted terms have large coefficients of oscillating signs, so that if one extrapolates further into weak coupling, the values diverge away from the expected 3 loop form until truly asymptotic values of the scale $1/a$ [4]. This is, of course, a symptom of the well-known fact that in the range where one currently performs calculations, no plausibly simple perturbative expression for $g_L^2(a)$ even begins to be adequate. This is unfortunate because it means that one cannot exploit dimensional transmutation in a simple way, to transform the value of the lattice bare coupling to a statement of how large the lattice spacing a is in units of the corresponding Λ parameter, which can then be expressed in terms of, say, $\Lambda_{\overline{MS}}$ in the theoretically and experimentally well-studied $\overline{MS}$ coupling scheme. (See [5] and [6] for recent reviews.)

This old problem has been approached both by stressing the need for improved lattice coupling schemes [7, 8] and the need for lattice spacing corrections [9]. Determining what is important is, however, rendered ambiguous by the fact that there is a smooth cross-over between strong and weak-coupling in SU(3), so one does not really know from which value of β it is appropriate to attempt a weak-coupling expansion in powers of g^2 and a.

In this paper we use the fact that for SU($N \geq 5$) there is a first order 'bulk' transition [10] separating the weak and strong coupling ranges, to remove the ambiguity of where one might expect a weak coupling expansion to be applicable. This enables us to quantify the importance of retaining $O(a^2)$ lattice corrections in addition to the usual continuum perturbative variation.

While our SU(6) and SU(8) calculations prove accurate enough to establish the need for $O(a^2)$ corrections in the relationship between a and $g^2(a)$, the range in a is not large enough to usefully constrain the coupling scheme and hence the value of $a\Lambda_{\overline{MS}}$. In fact experimental studies are not accurate enough either. (See for example figure10 in [11].) Fortunately there exists in SU(3) an accurate calculation of the running coupling in the 'Schrodinger functional' (SF) scheme that covers an energy range comparable to that of experiment, i.e. up to $\sim M_Z$, and with appreciably smaller errors [12, 13]. We shall use this scheme to obtain from the values of a/r_0 calculated in [3, 14] the continuum value of $r_0\Lambda_{\rm SF}$ and hence of $r_0\Lambda_{\overline{MS}}$. We compare this to what one obtains with some improved coupling extrapolations, and find that the Parisi mean-field improved coupling scheme [7] closely matches the SF result. We simultaneously perform a comparison with the SF scheme that does not involve the calculation of any physical quantity and therefore can be carried out to much weaker coupling. This also points to the 'goodness' of the mean-field scheme. Motivated by this we use this scheme for $N \neq 3$ to obtain continuum values for $\Lambda_{\overline{MS}}/\sqrt{\sigma}$ for all N, and in particular for $N \to \infty$.

In the next section we discuss the weak coupling behaviour of a on $g^2(a)$ in more detail, and establish our notation. We then move on to our calculations and then attempt to quantify the systematic errors on our final results. We end with a brief summary and our conclusions.

An abbreviated version of this work has been presented at Lattice 2007 [16]. This study forms part of a more detailed and extensive analysis that will appear elsewhere [17].

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2. The lattice running coupling

Ideally we would wish to be able to determine from the value of $g_L^2(a) \equiv 2N/\beta$ what is the lattice spacing a as expressed in units of the corresponding physical scale Λ_L (which can then be converted into a more familiar scale, such as $\Lambda_{\overline{MS}}$, using a one-loop calculation [18]). Although we know this to be possible in principle (dimensional transmutation) in practice doing so for the SU(3) gauge theory (and QCD) has proved notoriously ambiguous, despite the fact that we know the β-function in the lattice scheme to 3 loops [19]. There are some plausible reasons for this which will become apparent in the following discussion.

To establish how $g_L^2(a)$ is related to a we determine how a varies with $g_L^2(a)$. To do this we express a in units of some physical mass μ that we calculate in lattice units, i.e. as $a\mu(a)$. Now we know that different choices for the quantity μ will have different lattice spacing corrections:

$$\frac{a\mu'(a)}{a\mu(a)} = \frac{\mu'(0)}{\mu(0)}\left(1 + c_\sigma a^2\sigma(a) + O(a^4)\right) \tag{2.1}$$

where we have chosen to use the string tension σ to set the scale of these corrections. As an example, the ratio $m_{0^{++}}/\sqrt{\sigma}$, where $m_{0^{++}}$ is the lightest scalar glueball mass, has $c_\sigma \sim 2$ to 3 depending on N, while $T_c/\sqrt{\sigma}$, where T_c is the deconfining temperature, has $c_\sigma \sim 1/3$. Thus it is clear that the variation of $a\mu(a)$ with $g_L^2(a)$ *must* include a dependence on a that has the functional form shown in eq. (2.1), in addition to the variation expected for a running coupling in the continuum theory. Generically we should expect $c_\sigma = O(1)$ as illustrated by the examples just quoted. There is a similar dependence that follows from the fact that the β-function for $g_L^2(a)$ will have lattice spacing corrections

$$\frac{\partial g_L^2}{\partial \log a^2} = \beta_0 g_L^4 + \beta_1 g_L^6 + \beta_2^L g_L^8 + \cdots + O(a^2). \tag{2.2}$$

Here the β_i are the coefficients of the continuum β-function; β_0 and β_1 are scheme-independent while β_2^L requires a 2-loop calculation relating g_L^2 to a coupling for which β_2 is already known [19]. Thus, as has been emphasised in [9], the dependence of a on $g_L^2(a)$ will need to incorporate such lattice spacing corrections in addition to the perturbative expression that one obtains in the continuum. Using the string tension σ for our scale this leads to

$$\begin{aligned} a\sqrt{\sigma}(a) = \frac{\sqrt{\sigma}(0)}{\Lambda_s}\left(1 + c_\sigma^s a^2\sigma + O(a^4)\right) e^{-\frac{1}{2\beta_0 g_s^2}} \left(\frac{\beta_1}{\beta_0^2} + \frac{1}{\beta_0 g_s^2}\right)^{\frac{\beta_1}{2\beta_0^2}} \\ \times e^{-\frac{\beta_2^s}{2\beta_0^2} g_s^2 + \left(\frac{\beta_1\beta_2^s}{2\beta_0^3} - \frac{\beta_3^s}{4\beta_0^2}\right) g_s^4 + O(g_s^6)} \end{aligned} \tag{2.3}$$

where we make explicit the quantities that depend on the coupling scheme s being used (here $s = L$). We use the standard definition of the Λ parameter, where the constant term in the expansion of $1/g^2(a)$ is absorbed into the scale of the logarithm. In eq. (2.3) the terms that involve only β_0 and β_1 constitute the exact 2-loop continuum result. (That is to say, it is the exact result when $\beta_{j\geq 2} = 0$.) For higher orders we are not aware of any such neat closed form in terms of elementary functions, so we present their contribution

as a power series in g^2. Note also that although the coefficient c_σ^s is a power series in g_s^2, we shall, following usual practice, treat it as a constant in our fits, since g_s^2 does not vary very much in the region where the $O(a^2)$ correction is significant. Finally we remark that we could introduce the lattice corrections in different ways. A small change would be to use, say, the mass gap m_G rather than the string tension σ to set the scale for the $O(a^2)$ corrections in eq. (2.3). A more radical change would be to substitute for each occurence of $a^2\sigma$ on the r.h.s. of eq. (2.3) the expression provided by eq. (2.3) itself. After iteration this would transform the lattice spacing corrections from a power series in $a^2\sigma$ into a power series in the perturbative factor in eq. (2.3). All such variations are *à priori* equally valid.

In addition to these lattice spacing corrections, a further complication is that the $g_L(a)$ coupling scheme is expected to have large higher order perturbative corrections. This follows from the large ratio between Λ_L and $\Lambda_{\overline{MS}}$ [18]:

$$\frac{\Lambda_{\overline{MS}}}{\Lambda_L} = 38.853 \exp\left\{-\frac{3\pi^2}{11N^2}\right\} \tag{2.4}$$

This implies that in the relationship between the two couplings, $g^2_{\overline{MS}} = g_L^2(1+\gamma g_L^2 + O(g_L^4))$, the coefficient γ must also be large, since one can easily see that $\Lambda_{\overline{MS}}/\Lambda_L = \exp(\gamma/2\beta_0)$. So if $g^2_{\overline{MS}}$ is a 'good' scheme with modest higher order terms in the β-function, which is something we shall assume from now on (see [20]), this will almost certainly not be the case for g_L^2. This is confirmed by explicit calculation of the three-loop coefficient [19] where one finds

$$\beta_2^L \overset{N\to\infty}{\simeq} 6.9\beta_2^{\overline{MS}} \tag{2.5}$$

Since we only know the β-function to 3 loops, it would be wise to seek a lattice coupling scheme where there is less reason to expect large higher order corrections. This problem has of course been appreciated for a long time and there have been extensive efforts to improve the lattice coupling (for a review see [8].) Perhaps the simplest and oldest suggestion is the 'mean-field' improved coupling of Parisi [7]

$$\frac{1}{g_I^2} = \frac{1}{g_L^2}\left\langle \frac{1}{N}\mathrm{Tr}U_p \right\rangle \equiv \frac{u_p}{g_L^2} \tag{2.6}$$

which has a nice physical motivation as the effective coupling experienced by a background field (in a simple approximation) [7]. Since the expansion of $\langle \mathrm{Tr}U_p\rangle$ in g_L^2 is known to 3-loops [21] we can substitute

$$\frac{1}{g_I^2} = \frac{1}{g_L^2}\left\langle \frac{1}{N}\mathrm{Tr}U_p \right\rangle = \frac{1}{g_L^2}\left(1 - \omega_1 g_L^2 - \omega_2 g_L^4 - \omega_3 g_L^6 + O(g_L^8)\right) \tag{2.7}$$

into the 3-loop β-function for g_L^2 to obtain the 3-loop β-function for g_I^2, giving

$$\beta_2^I = \beta_2^L + \beta_0\omega_2 - \beta_1\omega_1 \overset{N\to\infty}{\simeq} 2.3\beta_2^{\overline{MS}} \tag{2.8}$$

which indeed promises smaller higher-order corrections than eq. (2.5). Similarly one finds

$$\frac{\Lambda_{\overline{MS}}}{\Lambda_I} = \frac{\Lambda_L}{\Lambda_I}\frac{\Lambda_{\overline{MS}}}{\Lambda_L} = \exp\left\{\frac{\omega_1}{2\beta_0}\right\} \times 38.853 \exp\left\{-\frac{3\pi^2}{11N^2}\right\} \simeq 2.633 \tag{2.9}$$

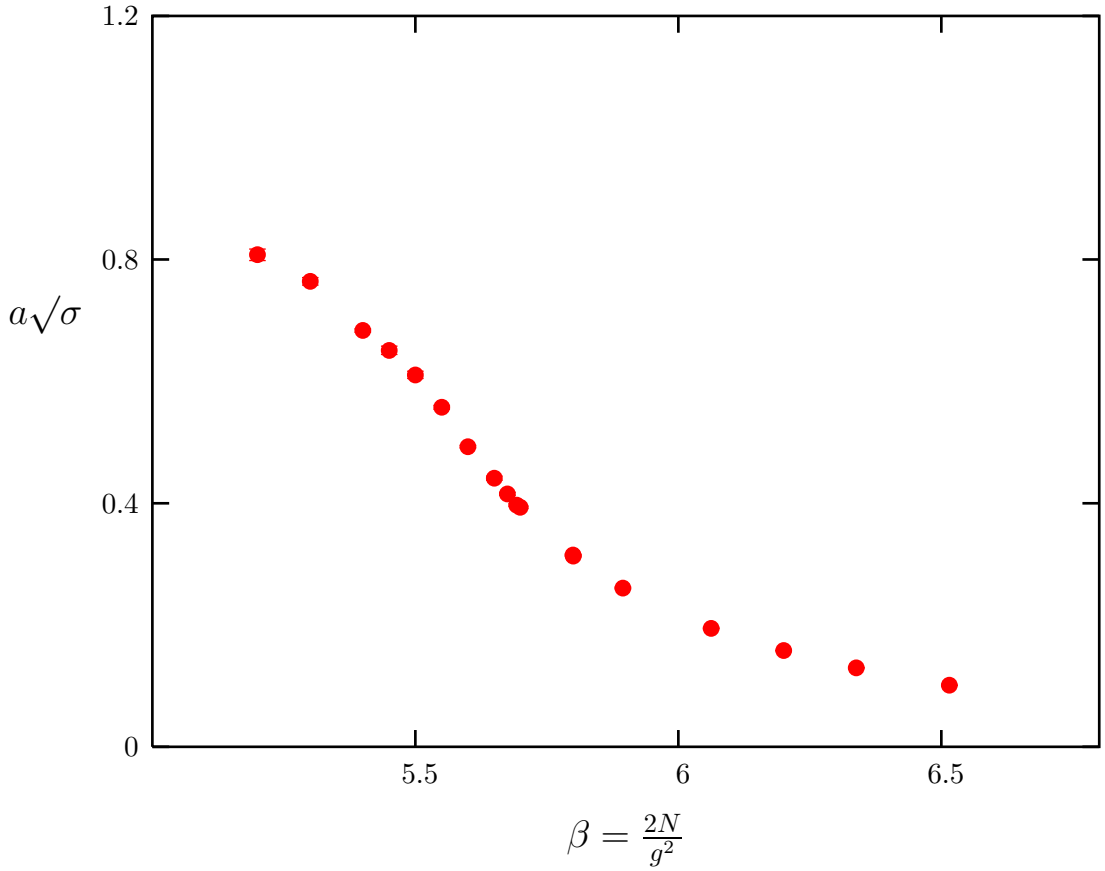

Figure 1: The SU(3) string tension versus the inverse lattice coupling, including the region of the crossover between strong and weak coupling.

using eq. (2.4) and $\omega_1 = (N^2-1)/8N$, which again indicates that the mean-field improved coupling scheme is a 'good' one. Motivated by this we shall focus on this coupling scheme, and some variations thereof, in our later calculations.

Although the need to improve the lattice coupling has long been recognised, the suggestion [9] that $O(a^2)$ lattice corrections are also needed has in practice been more controversial. As demonstrated above, there can be no question about the presence of such corrections. But one might question their importance. How plausible is this? As remarked above, we expect that generically $c_\sigma = O(1)$ in eq. (2.3), and since $a\sqrt{\sigma}(a)$ and other masses are typically calculated to a precision $\ll 1\%$, we would need $a^2\sigma \ll 0.01$ for the $O(a^2)$ correction to be negligible. In SU(3), for example, $a^2\sigma \simeq 0.01$ at $\beta = 6.5$. This coupling region is at the upper edge of the range in which useful calculations are usually performed. Thus generically we should expect $O(a^2)$ corrections to be important in current calculations. Of course it might have been that the coefficient c_σ just happened to be unexpectedly small, but the calculations in this paper will show that this is not the case.

Eq. (2.3) is only valid in weak coupling. This is as true of the lattice corrections as of the

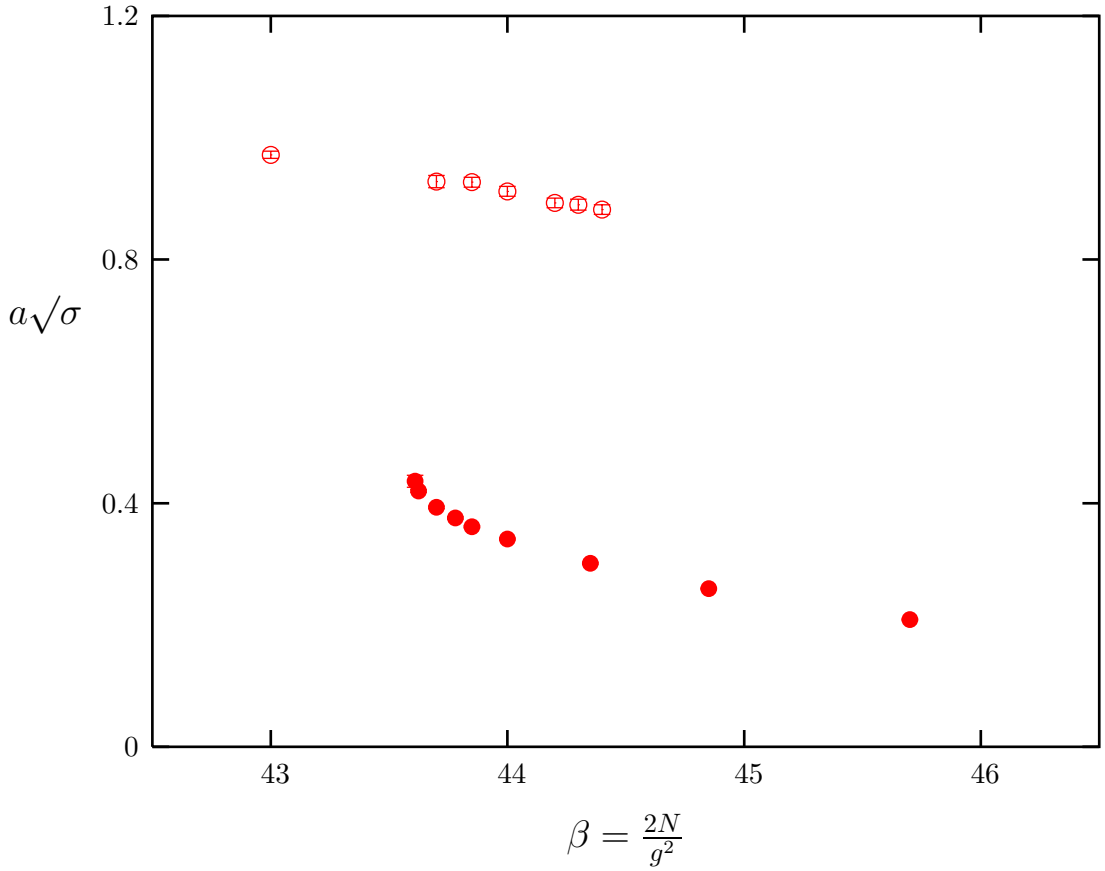

Figure 2: The SU(8) string tension versus the inverse lattice coupling, including the region of the first order 'bulk' transition between strong and weak coupling. Values ∘ are obtained coming from strong coupling, while the values • are obtained coming from weak coupling.

perturbative factor. In strong coupling the appropriate relationship is a very different one,

$$a^2\sigma \overset{g_L^2\to\infty}{=} -\log\frac{1}{g_L^2} + \sum_{n=0} c_n \left(\frac{1}{g_L^2}\right)^n . \tag{2.10}$$

(See [22] for a detailed review.) A problem that now arises for small N, and in particular for SU(3), is that there is a smooth cross-over between the strong and weak coupling regimes, with a mid-point that occurs close to the value of β where one typically starts to calculate interesting physical quantities. This is illustrated in figure 1 where we show how $a\sqrt{\sigma}$ varies with β for SU(3). The point of inflection, around $\beta \in [5.50, 5.60]$, provides an estimate of the mid-point of the cross-over. Clearly there must be some range of β beyond that where the functional form is still not pure weak coupling, but a priori we do not know how far that range extends. This complication disappears for $N \geq 5$ where the strong-weak coupling transition becomes a first order phase transition. This is illustrated in figure 2 for SU(8). Here we see a large hysteresis indicating a strong first order transition. The simultaneous large jump in the average plaquette, and hence action, shows this to be a 'bulk' transition where the physics changes on all length scales. It is reasonable to expect

that on the weak-coupling side of this bulk transition there are no strong-coupling artifacts. We shall take advantage of this feature by first testing our weak-coupling fit in eq. (2.3) to the values of $a\sqrt{\sigma}$ obtained on the whole weak coupling branch of the SU(8) theory, and then decreasing N to $N = 3$ (and $N = 2$) having established in the much less ambiguous large-N case what kind of fit is needed.

Our assumption that there are no strong-coupling artifacts on the weak-coupling side of the 'bulk' transition is not only plausible but is supported by analytic calculations in a related but simpler context: SU(N) lattice gauge theories in $1+1$ dimensions. Here (with the plaquette action) there is a strong-weak coupling cross-over at finite N that becomes a third-order phase transition at $N = \infty$ [30]. In this limit one can calculate the string tension as a function of β analytically [30] and one finds a simple expansion in powers of $1/\beta$ with no strong-coupling artifacts anywhere on the weak-coupling branch. While this does not prove that the same will be true in the case of $D = 3+1$, it provides an additional argument that such artifacts will be either absent or highly suppressed.

An interesting question concerns the functional form, for $N \leq 4$, in the extended cross-over region where the strong coupling expansion gradually transforms into a weak-coupling one. We can obtain some intuition from the $D = 1 + 1$ case where the whole problem reduces to a single SU(N) integral [30]. In particular, for SU(2) the running of the coupling becomes quite elementary,

$$u_p = \exp\{-a^2\sigma\} = \frac{\int_{-1}^{1} dz z(1-z^2)^{\frac{1}{2}} \exp(\beta z)}{\int_{-1}^{1} dz (1-z^2)^{\frac{1}{2}} \exp(\beta z)} \tag{2.11}$$

and one can analyse the behaviour in the cross-over region. We note that the terms in eq. (2.11) are obtained by applying $\partial/\partial\beta(1-\partial^2/\partial\beta^2)^{\frac{1}{2}}$ or $(1-\partial^2/\partial\beta^2)^{\frac{1}{2}}$ to

$$\int_{-1}^{1} dz \exp(\beta z) = \frac{1}{\beta}\left(e^{\beta} - e^{-\beta}\right). \tag{2.12}$$

So on the weak coupling side we will have, in addition to the expected weak coupling expansion in powers of $1/\beta$, a correction term that is an expansion in powers of $e^{-2\beta}$, whose coefficients are themselves series in powers of $1/\beta$. It is these exponential terms that are our strong-coupling artifacts. They arise from the compact integration range of the plaquette in eq. (2.11), and this general origin suggests that they will be present in some form in higher dimensions. This is most convincing for the average plaquette, where such $O(e^{-2\beta})$ corrections would create serious complications for the standard methods of extracting the gluon condensate in $D = 3 + 1$ since they (and their larger-N homologues) would dominate over the $O(a^4)$ condensate contribution. We note the close connection of these corrections to 'Z_N vortex-instantons' in $D = 1+1$. In $D = 2+1$ these will generalise to Z_N 'monopole-instantons' and in $D = 3 + 1$ to Z_N 'monopoles', as well as Z_N vortex lines and sheets. Such ultraviolet fluctuations will disorder small Wilson loops, and will affect the string tension in strong coupling. Around the cross-over one might expect some complicated contribution that is the ratio of powers and exponentials in β. While it is not easy to be precise about this, it is clear that the functional form suggested by our

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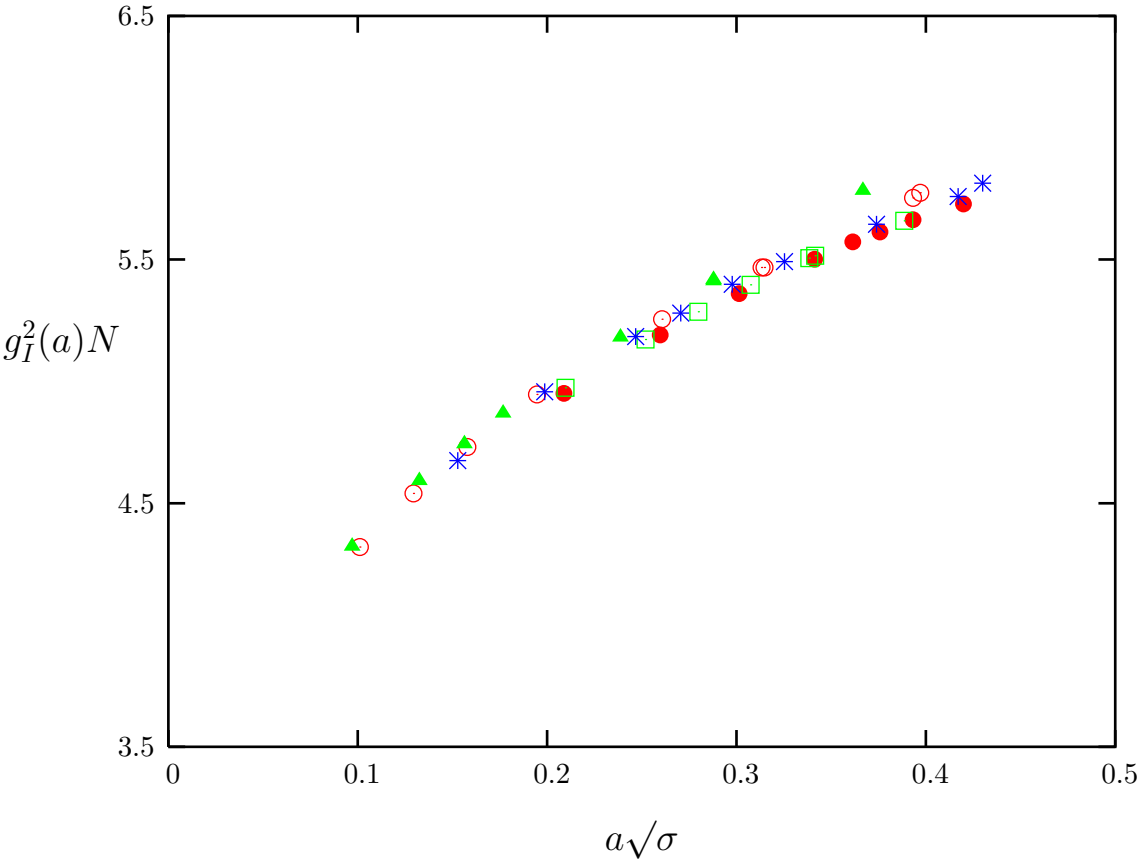

Figure 3: The (mean-field improved) bare 't Hooft coupling as a function ot the scale in units of the calculated string tension, for $N = 2$ ($\triangle$), $N = 3$ ($\circ$), $N = 4$ ($*$), $N = 6$ ($\square$), and $N = 8$ ($\bullet$).

$D = 1 + 1$ example is very different to anything one might imagine on the basis of weak coupling arguments, and it will be no surprise if we find that at small N we have to go further into weak coupling to be able to apply standard weak coupling fits.

Of course, using SU(N) gauge theories at larger N to teach us something about SU(3) is only convincing if it is clear that SU(3) is 'close to' SU(∞). That this is so in the context of the bare coupling was demonstrated in figure7 of [10] using calculations for $2 \leq N \leq 5$. Here we repeat the exercise with calculations that not only go to larger N, but also to smaller values of a at smaller N. So, for each value of β at which we calculate the string tension (see section 3.1) we extract the bare mean-field improved 't Hooft coupling $g_I^2(a)N$ using eq. (2.6). We then plot it against the length scale a on which it is running, with a expressed in units of the string tension. Since the physics has a smooth large-N limit, this is a common unit up to corrections of $O(1/N^2)$ which we expect to be modest, since this is what one finds for various mass ratios [10].

The resulting plot is shown in figure 3. This provides convincing evidence that the weak-coupling running of the bare coupling is in fact very similar for all values of N. Thus it makes sense to extract lessons in this context from larger N for all SU(N) gauge theories.

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SU(2)			
β	L	am_l	u_p
2.1768	8	1.990(64)	0.56122
2.2400	8	1.380(16)	0.58286
2.2986	10	1.240(13)	0.60180
2.3715	12	0.9071(94)	0.62272
2.3726	12	0.9088(63)	0.62302
2.4265	16	0.8470(68)	0.63632
2.5115	20	0.5728(54)	0.65421
2.5500	20	0.4363(40)	0.66137
2.6000	24	0.3778(44)	0.67001
2.7000	32	0.2688(30)	0.68557

Table 1: The mass, am_l, of a closed flux loop of length L, and the average plaquette, u_p, at the indicated values of β in SU(2).

3. The calculation

3.1 String tensions and r_0

The string tensions used in this paper have been obtained from calculations of the ground state energies of confining flux loops wrapped around a spatial torus. We use the results of [23] supplemented in many cases by either higher statistics at the same values of β or new calculations at both higher and lower values of β. In tables 1–5 we list these masses, am_l, together with the length, L, of the flux loop in each case, and the average plaquette, u_p, which will be needed for transforming to the mean-field coupling scheme.

To extract the string tension we use

$$am_l(L) = a^2\sigma L - \frac{\pi}{3L}. \tag{3.1}$$

The linear term has been corrected by the universal string correction [24] with a coefficient that corresponds to a bosonic string theory (the universality class of the Nambu-Goto string). That this is in fact the appropriate universality class for the confining flux tube in D=3+1 non-Abelian gauge theories, has received support from many lattice calculations in both SU(2) and SU(3) (see for example [25]) as well as at larger N [26]. The string lengths in tables 1–5 have been chosen to be large enough, $aL\sqrt{\sigma} \geq 3$, that the string correction in eq. (3.1) provides only a $\sim 5\%$ shift in the calculated value of $a\sqrt{\sigma}$, so higher order corrections in $1/a^2\sigma$ (the natural expansion parameter in the relationship between $am_l(L)$ and $a^2\sigma$) should be very small indeed. We shall estimate the systematic error that this brings about below.

In the case of SU(3), the calculated values of the Sommer parameter, r_0, extend considerably further into weak coupling than the string tension. For this reason we will supplement our fits of $a\sqrt{\sigma}$ with those of a/r_0, as listed in table 6. These have been taken from table 2.6 of [14] where, for the values at higher β, we have translated from r_c to r_0 using

SU(3)			
β	L	am_l	u_p
5.6500	8	1.425(25)	0.53750
5.6750	8	1.249(15)	0.54366
5.6925	8	1.130(12)	0.54756
5.6993	8	1.106(11)	0.54896
5.7995	10	0.8860(79)	0.56755
5.8000	10	0.8766(86)	0.56764
5.8945	12	0.7283(67)	0.58111
6.0625	16	0.5408(46)	0.60034
6.2000	20	0.4465(32)	0.61362
6.3380	24	0.3588(29)	0.62560
6.5150	32	0.2943(35)	0.63948

Table 2: As in table 1 but for SU(3).

SU(4)			
β	L	am_l	u_p
10.480	8	1.348(15)	0.5253
10.500	8	1.234(13)	0.52920
10.550	8	0.9878(87)	0.53732
10.590	10	1.1296(94)	0.54259
10.635	10	0.9587(105)	0.54764
10.637	10	0.9541(43)	0.54789
10.700	10	0.7789(90)	0.55408
10.789	12	0.7885(56)	0.56169
10.870	12	0.6456(72)	0.56793
11.085	16	0.5663(54)	0.58239
11.400	20	0.0044(44)	0.60044

Table 3: As in table 1 but for SU(4).

eq. (2.73) therein. (We note from figure2.4 of [14] that the quoted error on the conversion factor appears to encompass any possible dependence on a.) We remark that the lower β values are from [3] and the higher values from [15].

We have calculated the values of the action listed in table 6. These calculations have been performed on 8^4 lattices and will therefore differ slightly from the large volume limit. However we have checked, performing a calculation on 6^4 lattices for the values of β in table 2, that these minute finite volume corrections to the action, lead to errors that are negligible compared to the relevant statistical errors.

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SU(6)			
β	L	am_l	u_p
24.300	8	1.246(14)	0.51721
24.350	8	1.077(11)	0.52254
24.425	10	1.210(12)	0.52810
24.500	10	1.063(12)	0.53258
24.515	10	1.041(11)	0.53340
24.670	10	0.8431(89)	0.54089
24.845	12	0.8540(81)	0.54816
25.050	12	0.6686(60)	0.55570
25.452	16	0.6396(55)	0.56866

Table 4: As in table 1 but for SU(6).

SU(8)			
β	L	am_l	u_p
43.625	8	1.279(15)	0.51220
43.70	8	1.106(12)	0.51713
43.78	8	0.9989(94)	0.52094
43.85	8	0.9139(78)	0.52374
44.00	10	1.0599(91)	0.52879
44.35	10	0.8036(77)	0.53848
44.85	12	0.7222(61)	0.54980
45.70	16	0.6326(45)	0.56571

Table 5: As in table 1 but for SU(8).

SU(3) ; r_0		
β	r_0/a	u_p
5.70	2.922(9)	0.54939
5.80	3.673(5)	0.56778
5.95	4.898(12)	0.58846
6.07	6.033(17)	0.60158
6.20	7.380(26)	0.61384
6.40	9.740(50)	0.63085
6.57	12.18(10)	0.64365
6.69	14.20(12)	0.65204
6.81	16.54(13)	0.65998
6.92	19.13(15)	0.66685

Table 6: The value of r_0 [14], and the average plaquette, u_p, at the indicated values of β in SU(3).

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N	$a\sqrt{\sigma} \in$	$\sqrt{\sigma}/\Lambda_I$	c_σ	$\chi^2/n_{\rm df}$
2	[0.177,0.097]	4.566(28)	3.83(26)	1.8
3	[0.261,0.101]	4.888(17)	2.08(10)	1.1
4	[0.374,0.153]	5.005(20)	1.52(5)	1.0
6	[0.415,0.210]	5.131(25)	1.30(5)	0.34
8	[0.420,0.209]	5.199(21)	1.18(4)	0.27

Table 7: Results of weak-coupling fits using eq. (3.2) over the indicated ranges of $a\sqrt{\sigma}$ and for various N.

SU(8)					
s	$a\sqrt{\sigma} \in$	$\sqrt{\sigma}/\Lambda_s$	c_σ	$\chi^2/n_{\rm df}$	$\Lambda_{\overline{MS}}/\sqrt{\sigma}$
latt	[0.420,0.209]	96.80(47)	3.71(5)	0.37	0.3848(19)
u_p 1-loop	[0.420,0.209]	8.980(43)	3.88(5)	0.54	0.2932(14)
u_p 2-loop	[0.420,0.209]	6.832(43)	3.67(5)	0.38	0.3854(25)
u_p 3-loop	[0.420,0.209]	6.100(28)	3.50(5)	0.32	0.4316(20)
u_p meas	[0.420,0.209]	5.199(21)	1.18(4)	0.27	0.5064(21)

Table 8: Weak-coupling fit using eq. (3.2) to the whole weak-coupling branch in SU(8); for various coupling schemes s.

3.2 Weak-coupling fits at all N

We begin by fitting the values of the SU(8) string tension that have been calculated on the weak coupling branch in figure 2. We attempt to fit all the way to the point at which the system tunnels through to the strong coupling phase. Although at lower β part of this branch is metastable, the barrier to the strong coupling phase is so large that its metastability is irrelevant. One can see this from the fact that the distribution of values of the plaquette (averaged over the whole volume for a given gauge field) shows no sign of a tail developing towards the very different values that it would take in the strong coupling phase.

Working in the mean-field improved coupling scheme $s = I$, we find that we can obtain an excellent fit to all the values in table 5, using the $O(a^2)$, 3-loop truncation of eq. (2.3)

$$a\sqrt{\sigma}(a) = \frac{\sqrt{\sigma}(0)}{\Lambda_I}\left(1 + c_\sigma^I a^2\sigma\right) e^{-\frac{1}{2\beta_0 g_I^2}} \left(\frac{\beta_1}{\beta_0^2} + \frac{1}{\beta_0 g_I^2}\right)^{\frac{\beta_1}{2\beta_0^2}} e^{-\frac{\beta_2^I}{2\beta_0^2} g_I^2}. \tag{3.2}$$

The best fit is illustrated in figure 4 and the fitted values of $\sqrt{\sigma}/\Lambda_I$ and the constant c_I are given in table 7. We see that $c_I = 1.18 \pm 0.04$ is indeed $O(1)$ as naively expected. In fact any attempt to fit without a lattice correction, i.e. with $c_I = 0$, fails very badly, even if we include in our fit only the values of $a\sqrt{\sigma}$ at the weakest couplings. Thus for $N = 8$ it is clear that the lattice spacing corrections are important and once taken into account they allow a very good fit with the usual 3-loop perturbative running coupling. This turns out to be equally true for SU(6), and we list the fitted parameters in table 7.

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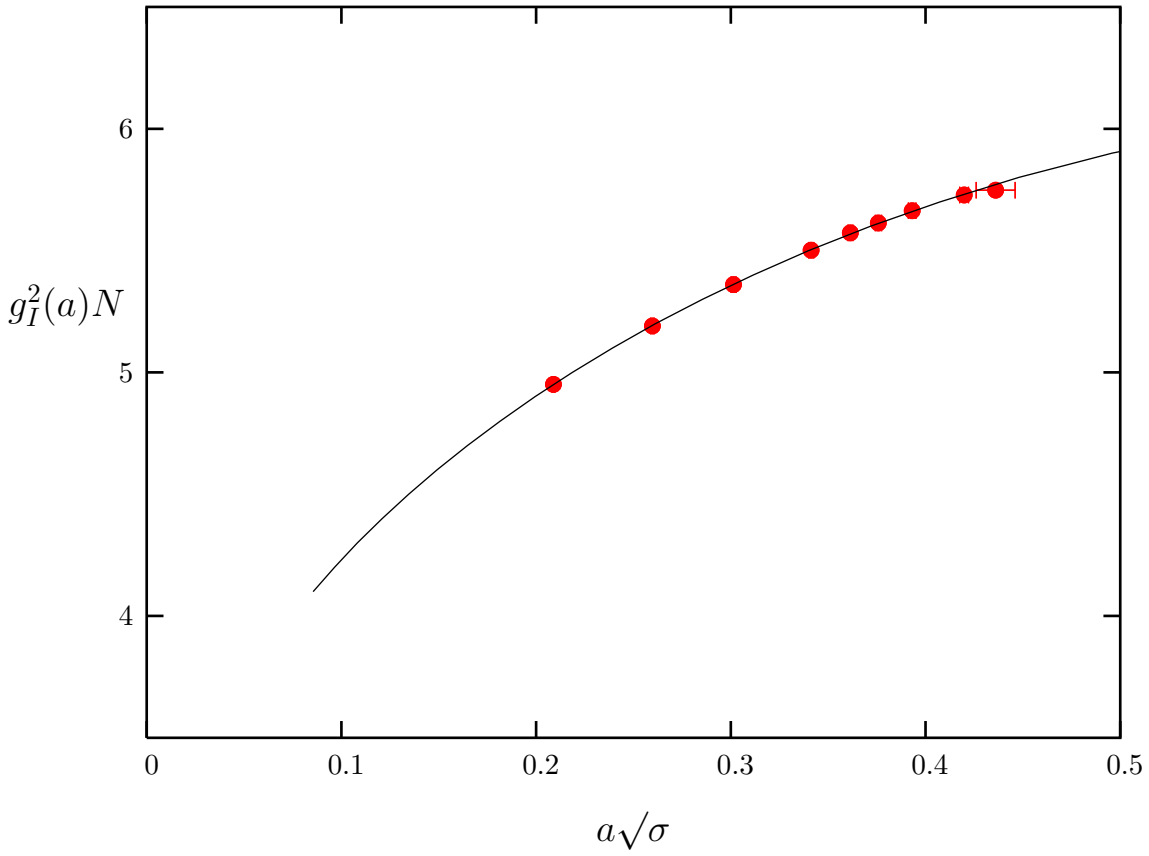

Figure 4: The 't Hooft coupling, defined from the mean-field improved lattice bare coupling, eq. (2.6), as a function of the scale a in SU(8). Shown is the 3-loop perturbative running modified by a $O(a^2)$ lattice correction.

Having established at these larger N that lattice spacing corrections are indeed needed, we proceed to lower N using the same functional form as in eq. (3.2). For SU(4), where the bulk transition has weakened to a relatively sharp cross-over [10], a good fit with eq. (3.2) is still possible but only if we restrict ourselves to somewhat weaker couplings. This is displayed in table 7 where we express the fitted range in terms of $a\sqrt{\sigma}$ so as to provide a common physical measure of comparison for different N. In SU(3) the bulk cross-over becomes smoother and we have to move even further into weak coupling to obtain an acceptable fit. This is illustrated in figure 5. Finally for SU(2) the bulk crossover is smoother still and we find that even if we include only the values at weakest coupling, we obtain an uncomfortably large χ^2 per degree of freedom, as shown in table 7. All this accords with the naive expectation that the smoother the strong-to-weak coupling crossover, the further into weak-coupling one has to go before weak-coupling expansions become applicable.

The computational cost of SU(N) lattice calculations grows roughly $\propto N^3$, and so our SU(3) calculations extend to much smaller values of a than for SU(8). One might wonder

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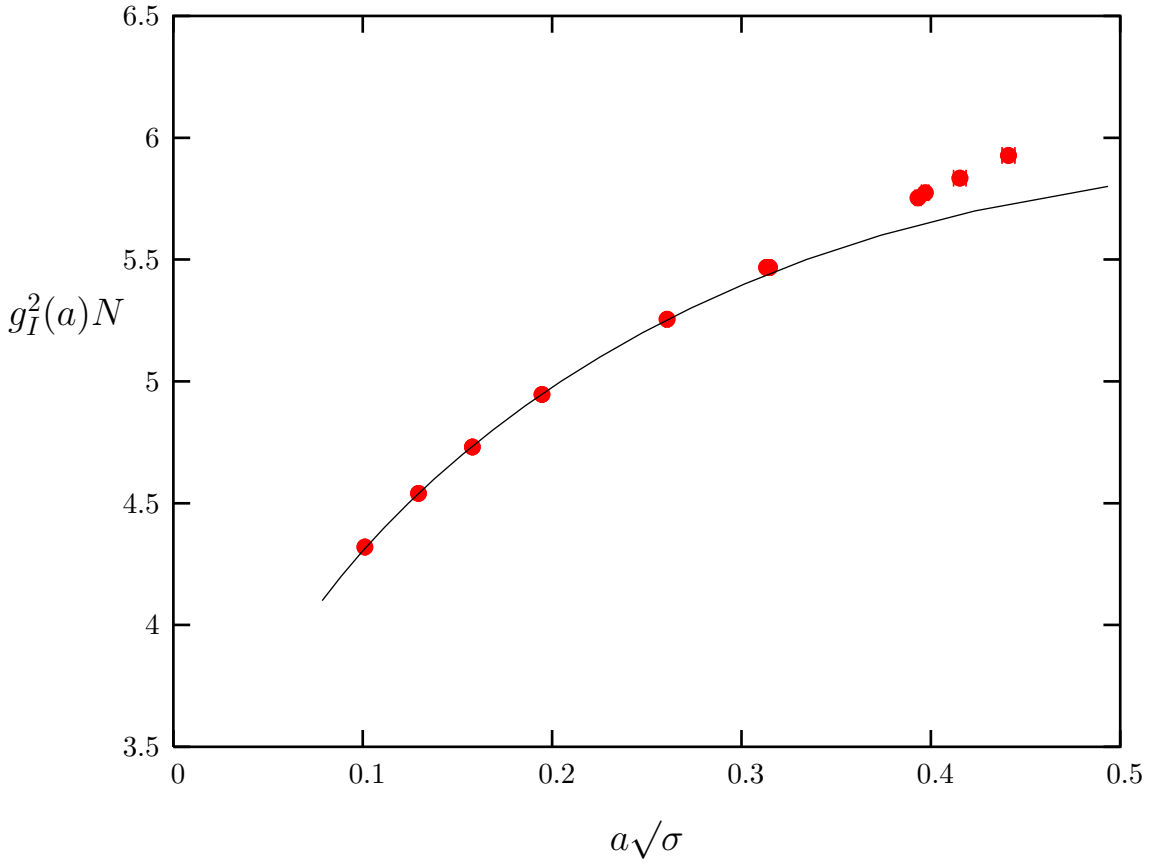

Figure 5: The 't Hooft coupling, defined from the mean-field improved lattice bare coupling, eq. (2.6), as a function of the scale a in SU(3). Shown is the 3-loop perturbative running modified by a $O(a^2)$ lattice correction.

if it is this greater range that makes the fit to SU(3) so much more difficult than for SU(8). In fact this is not the case. If we fit the SU(3) values over a range of $a\sqrt{\sigma}$ values that is very similar to that of the whole SU(8) range, we obtain a statistically unacceptable $\chi^2/n_{\rm df} \sim 8$.

We can now take each value of $\sqrt{\sigma}/\Lambda_I$ in table 7 and transform the ratio to $\sqrt{\sigma}/\Lambda_{\overline{MS}}$ using eq. (2.9). If we then extrapolate to $N = \infty$ using the expected leading $O(1/N^2)$ correction, we obtain a good fit for $N \geq 3$, giving

$$\frac{\Lambda_{\overline{MS}}}{\sqrt{\sigma}} = 0.503(2)(?) + \frac{0.33(3)(?)}{N^2} \qquad ; \qquad N \geq 3. \tag{3.3}$$

The final result for $\Lambda_{\overline{MS}}$ is remarkably precise, but this appearance is quite misleading since the quoted error is only statistical. The systematic errors, indicated by the '?' in eq. (3.3) are potentially very much larger. The most significant uncertainty, the choice of coupling scheme, is what we address first. There are some further smaller systematic errors (which are still much larger than the statistical errors) which will be discussed later on in this section.

SU(3): SF scheme						
interp.	p	$\beta \in$	$1/r_0\Lambda_{\rm SF}$	c_r	d_r	$\chi^2/n_{\rm df}$
latt	2	[6.257,6.9079]	3.245(24)	1.64(80)	1.05(26)	0.52
u_p 2-loop	2	[6.257,6.9079]	3.243(20)	1.61(64)	1.10(28)	0.54
u_p 3-loop	2	[6.257,6.9079]	3.238(23)	1.78(63)	1.10(28)	0.38
u_p meas	2	[6.257,6.9079]	3.212(27)	2.60(55)	1.13(28)	0.25
u_p 2-loop	1	[6.257,6.9079]	3.169(23)	1.63(44)	0.34(8)	0.45
u_p meas	1	[6.257,6.9079]	3.141(40)	2.21(58)	0.36(09)	0.18
u_p meas	2	[6.257,7.2611]	3.215(17)	2.64(45)	1.09(23)	0.28

Table 9: Weak coupling fits to r_0 in SU(3) using the SF coupling scheme, as in eq. (3.5).

One might have hoped hope that the accuracy of our calculation of $a\sqrt{\sigma}$ would prove sufficient to usefully constrain what are the best coupling schemes to use. Unfortunately this turns out not to be the case. To show this most graphically we consider three alternative schemes which all have the same scale $\Lambda_s = \Lambda_I$. These are obtained by replacing the value of the average plaquette u_p in eq. (2.6) by a truncation of its perturbative expansion in eq. (2.7) to either $O(g^2)$, or $O(g^4)$, or $O(g^6)$. We label these schemes by $s = I_1, I_2, I_3$ respectively; i.e.

$$\frac{1}{g^2_{I_j}} = \frac{1}{g^2_L}\left(1 - \omega_1 g^2_L - \cdots - \omega_j g^{2j}_L\right) \tag{3.4}$$

using the notation of eq. (2.7). For $j \geq 1$ these schemes are identical to the mean-field scheme, $s = I$, to $O(g^2)$, so they will have exactly the same Λ_s parameter. We now take eq. (3.2) and fit our SU(8) string tensions, replacing the values of g^2_I by the corrsponding values of $g^2_{I_i}$ for $i = 1, 2, 3$. We obtain perfectly good fits in all cases as shown in table 8. However, as we see there, the fitted values of $\sqrt{\sigma}/\Lambda_I$ vary by almost a factor of 2 between these four coupling schemes. (For completeness we show the fit using the straightforward lattice coupling scheme $s = L$, which, when translated to a value of $\sqrt{\sigma}/\Lambda_{\overline{MS}}$, is similar to the fit in the I_2 scheme.) We obtain very similar results for $N = 6$. Thus in the case of larger N, where the presence of a first order bulk transition defines for us where a weak coupling expansion should begin to be applicable, there is in practice no useful constraint on the coupling scheme. At smaller N, where there are in fact marked differences in where one can begin fitting with different schemes, the presence of a smooth strong-to-weak crossover means that it is not clear what significance one should read into any particular comparison.

3.3 SU(3): choosing a scheme

The simplest way to determine if any given bare coupling scheme is 'good' is to compare it against a running coupling that has been calculated on the lattice with enough statistical and systematic precision to serve as a benchmark. A good example is the Schrodinger Functional (SF) scheme of the Alpha Collaboration [27]. In [13] the SU(3) calculation [12] has been extended so that it covers a range of energy scales comparable to that covered by experimental measurements, but with much greater precision. (Compare figure4 of [13] with

SU(3) :						
s	$a/r_0 \in$	$1/r_0\Lambda_s$	c_r	$\chi^2/n_{\rm df}$	$r_0\Lambda_{\overline{MS}}$	$r_0\sqrt{\sigma}$
latt	[0.166,0.052]	53.26(21)	7.03(20)	1.26	0.5409(22)	1,143(11)
u_p 1-loop	[0.103,0.052]	5.883(23)	10.65(30)	0.89	0.4475(18)	1.166(11)
u_p 2-loop	[0.166,0.052]	4.891(20)	6.93(20)	1.28	0.5383(22)	1.156(14)
u_p 3-loop	[0.204,0.052]	4.567(16)	5.87(13)	0.85	0.5765(20)	1.159(7)
u_p meas	[0.204,0.052]	4.215(16)	3.00(13)	0.48	0.6246(24)	1.160(6)

Table 10: Weak coupling fits to r_0 in SU(3) using the indicated coupling schemes, s, in appropriate modifications of eq. (3.2).

figure10 of [11].) Its (continuum) running is well described by its 3-loop β-function [12, 13], while the size of the non-universal β_2 term is known to be modest [28, 13]. All this encourages us to use it as our benchmark good coupling scheme.

The SF coupling is a coupling defined on a length scale la for a given a, where l can take any integer value. (Restrictions arise in practice.) The values of β and l at which calculations exist, with the corresponding values of $g^2_{\rm SF}(al)$, are listed in tables 3 and 6 of [13]. Our strategy is to take some physical quantity μ that has been calculated over a large range of lattice couplings, interpolate from the values of β at which it has been calculated to the values of β at which $g^2_{\rm SF}(al)$ has been calculated for some l, and then fit these values using a 3-loop perturbative expression, modified by the expected lattice corrections, to obtain $\mu/\Lambda_{\rm SF}$. (Note that in contrast to extrapolation, interpolation does not generate significant systematic errors.) We then fit these same values of μ using eq. (3.2) with various improved lattice coupling schemes to obtain alternative estimates of μ/Λ_s and hence of $\mu/\Lambda_{\rm SF}$. If the latter is consistent with the value obtained using $g^2_{\rm SF}(al)$, we have evidence that the corresponding 'improvement' is indeed a 'good' one.

Our string tension calculations in table 2 do not have enough overlap in β with the $g^2_{\rm SF}(la)$ calculations to be useful for this purpose. (Within the relevant range $\beta \leq 6.515$, values of $g^2_{\rm SF}$ have been calculated at only two values of β.) Instead we turn to the r_0 calculations in table 6 which do have a useful overlap. We fit using

$$\frac{la}{r_0(a)} = \frac{1}{r_0\Lambda_{\rm SF}}\left(1 + c_r^{\rm SF}\frac{a^2}{r_0^2} + d_r^{\rm SF}\frac{1}{l^p}\right) \times e^{-\frac{1}{2\beta_0 g^2_{\rm SF}(la)}}\left(\frac{\beta_1}{\beta_0^2} + \frac{1}{\beta_0 g^2_{\rm SF}(la)}\right)^{\frac{\beta_1}{2\beta_0^2}} e^{-\frac{\beta_2^{\rm SF}}{2\beta_0^2}g^2_{\rm SF}(la)}. \tag{3.5}$$

Here there are two lattice spacing corrections. The usual $O(a^2)$ term arises from corrections to $r_0(a)$ etc. while the $O(1/l^p)$ term arises from lattice corrections to $g^2_{\rm SF}(la)$ on the scale $l \times a$. As indicated in [12, 13] one expects the leading correction for the SF scheme to have $p = 1$. However the fact that the coupling is Symanzik-improved means that the dominant correction in our range of a might in fact have $p = 2$ [12, 13]. We shall use fits with both values of p taking the difference as part of our estimate of the systematic error. We expect

(as usual) that $c_r \sim O(1)$ but we anticipate that d_r will be small given that very small scaling violations are seen in the step-scaling function in [12, 13].)

To obtain as strongly constrained a fit as possible, we want to maximise the number of values of the running coupling that we fit in the range $\beta \in [5.70, 6.92]$ where we have calculations of a/r_0. We therefore use not only the values of $g^2_{\rm SF}(la)$ and $g^2_{\rm SF}(2la)$ in table 3 of [13] but also the values of $g^2_{\rm SF}(la)$ listed in table 6 therein. (We exclude the last 4 rows of table 6 since they use a different improvement.) We list in table 9 the results of our various fits. We show separately the results of using different interpolations in obtaining $a/r0$. As expected we find that this makes a negligible difference. We show fits for powers $p = 1$ and $p = 2$ and see that this makes a small $\sim 2\%$ difference. Finally we give an example of a fit that uses values in the additional range $\beta \in [6.92, 7.26]$ where we obtain the values of a/r_0 by the less reliable process of extrapolation. Again there is only a small change in the fit.

We note that the region of couplings $\beta \in [6.22, 6.91]$ that we actually use in our fit, corresponds to the range of scales

$$\frac{\mu}{\Lambda_{\rm SF}} = \frac{1}{a\Lambda_{\rm SF}} = \frac{1}{r_0\Lambda_{\rm SF}} \times \frac{r_0}{a} \in [23.9, 58.6]. \tag{3.6}$$

If we look at figure4 in [13] we see that in this range, the 3-loop formula we use already appears to provide a very good approximation to the continuum β-function. The lattice spacing corrections are less well determined but by taking the range of results spanned by both the $p = 1$ and the $p = 2$ fits, our estimate of this systematic error should be credible. Putting all this together we obtain

$$\frac{1}{r_0\Lambda_{\rm SF}} = 3.2(1) \longrightarrow r_0\Lambda_{\overline{MS}} = 0.640(20) \tag{3.7}$$

using $\Lambda_{\rm SF} \simeq 0.48811\Lambda_{\overline{MS}}$ [12]. If we now compare this with the values in table 10 that have been obtained by fitting r_0/a using eq. (3.2) with various bare coupling schemes, we see that the Mean-Field scheme produces values that are consistent. It is therefore plausible to adopt the latter as our 'good' scheme, while incorporating a systematic error $\sim 6\%$ based on the difference between the value in table 10 and the value ~ 0.66 that one gets at one standard deviation in eq. (3.7).

As a by-product of these calculations we also obtain an updated value for the relationship between the two standard scales r_0 and $\sqrt{\sigma}$:

$$r_0\sqrt{\sigma} = 1.160(6)(6) \quad : \quad \mathrm{SU}(3). \tag{3.8}$$

The first error is statistical and the second is systematic. It includes a $\sim 0.5\%$ error on the value of σ and a $\sim 0.2\%$ error from $O(a^4)$ corrections. (See section 3.5.) We cannot estimate, and therefore neglect, any systematic error on r_0. Note that there are no perturbative uncertainties here, as we see from the similarity of the various estimates of $r_0\sqrt{\sigma}$ in table 10. What we are doing is equivalent to interpolating a/r_0 and $a\sqrt{\sigma}$ to common values of β, taking the ratio, and performing a continuum extrapolation with conventional lattice corrections. This process is insensitive to the precise interpolation as long as it is smooth.

3.4 Comparing schemes directly

Here we introduce a method for comparing different coupling schemes directly, without the use of any physical quantity such as r_0/a or $a\sqrt{\sigma}$. This has the great advantage that it allows us to perform comparisons much deeper into weak coupling.

For a scheme s define the 3-loop perturbative factor

$$F_3^s[g_s^2] = e^{-\frac{1}{2\beta_0 g_s^2}} \left(\frac{\beta_1}{\beta_0^2} + \frac{1}{\beta_0 g_s^2}\right)^{\frac{\beta_1}{2\beta_0^2}} e^{-\frac{\beta_2^s}{2\beta_0^2} g_s^2}. \tag{3.9}$$

Now we expect for the SF scheme

$$la\Lambda_{\rm SF} \simeq \left\{1 + \frac{c_1}{l^p}\right\} F_3^{\rm SF}[g^2_{\rm SF}(al)] \tag{3.10}$$

and for a 'good' improved scheme I' based on the bare lattice coupling

$$a\Lambda_{I'} \simeq \left\{1 + c'a^2\right\} F_3^{I'}[g^2_{I'}(a)], \tag{3.11}$$

up to the various higher order corrections. If we now replace the a^2 on the r.h.s. of eq. (3.11) by the expression for a in eq. (3.10), and if we then take the ratio of the two equations, we obtain

$$\frac{\Lambda_{\rm SF}}{\Lambda_{I'}} \simeq c_0 = \frac{1}{l}\frac{F_3^{\rm SF}[g^2_{\rm SF}(al)]}{F_3^{I}[g^2_{I'}(a)]} \frac{\left\{1 + \frac{c_1}{l^p}\right\}}{\left\{1 + c_2\frac{1}{l^2}\left\{1 + \frac{c_1}{l^p}\right\}^2 \{F_3^{\rm SF}[g^2_{\rm SF}(al)]\}^2\right\}} \tag{3.12}$$

where $c_2 = c'/\Lambda^2_{\rm SF}$. We can now perform a fit for the constants c_0, c_1 and c_2 over ranges of β further and further into weak coupling, and see how rapidly c_0 approaches the known value of $\Lambda_{\rm SF}/\Lambda_{I'}$. The more rapidly it does so, the 'better' we may judge the coupling scheme to be.

Before proceeding, some remarks. We do not expect exact agreement between the fitted and known values of $\Lambda_{\rm SF}/\Lambda_{I'}$ because we are missing the 4-loop and higher contributions to the β-function. Indeed it is precisely the discrepancy that will tell us how 'good' is our I' scheme. In addition, over the wider range of $g^2(a)$ values that becomes accessible with this method, we may well need to worry about the fact that our supposed constants, c_1 and c_2, are in fact power series in g^2. These, and other issues that arise in such a calculation, we shall ignore, with the caution that one should therefore regard our calculation as being less than fully quantitative. On the other hand, we note that the $O(a^2)$ corrections to eqs. (3.10), (3.11) are in fact negligible in such a comparison except at the very smallest values of β, thus largely eliminating one source of systematic error.

We have grouped the calculated values of $g^2_{\rm SF}$ by the ranges of β within which they have been calculated, and we have fitted each such group of values separately. Each group has ~ 10 values of $g^2_{\rm SF}$ which is more than enough to constrain the three parameters in the fit. We can thus see how the fitted value of c_0 compares to the ratio of the Λ parameters as we go further into weak coupling. We do so separately for the mean-field coupling, $I' = I$, for the variation $I' = I_3$ where the plaquette is replaced by its 3-loop approximation, and finally for the lattice bare coupling, $I' = L$. We choose to take a power $p = 2$ although in practice we get much the same picture with $p = 1$.

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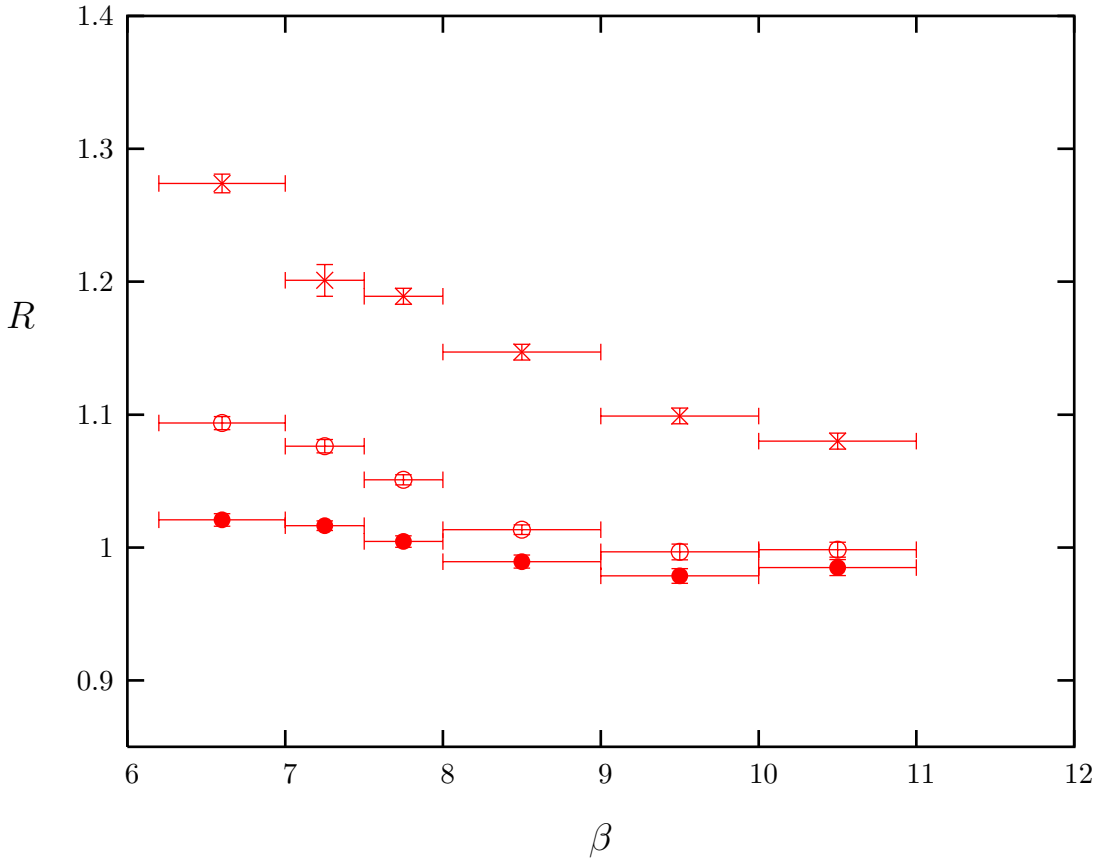

Figure 6: Calculated values of $R = c_0/(\Lambda_{\rm SF}/\Lambda_{I'})$, where c_0 comes from the fit in eq. (3.12). For the $I' = I$, $\bullet$, $I' = I_3$, $\circ$, and the $I' = L$, $\times$, lattice coupling schemes.

The results of our fits are shown in figure 6. We plot there the ratio of c_0 to its asymptotic value, i.e. $\Lambda_{\rm SF}/\Lambda_I$ for schemes I and I_3, and $\Lambda_{\rm SF}/\Lambda_L$ for the lattice scheme. We see quite clearly that Parisi's original mean-field scheme, using the full plaquette in the improvement, is consistent with converging remarkably quickly to the expected value (with a small deviation that is consistent with a slowly decreasing 4-loop term). If the SF coupling is 'good', then so, it would appear, is the mean-field scheme. The other schemes considered fare much less well by this criterion.

A final aside. We have emphasised that the SF coupling possesses various desirable properties — it has been calculated over a very large range of scales, it is very precise, it has modest higher order perturbative corrections (as indicated by the results of [12, 13]). All this makes it an attractive benchmark coupling. However we also recall that $g^2_{\rm SF}(l)$ is defined on a l^4 torus. Now, it is known that SU($N = \infty$) gauge theories suffer a sequence of phase transitions as $l \to 0$ (see [31] and references therein). Each of these transitions involves the breaking of a Z_N symmetry in one of the space-time directions. The first transition is the usual deconfining transition and the remaining ones can be interpreted as continuations of deconfining transitions on ever more dimensionally reduced space-time

volumes [32]. At finite N these phase transitions will become cross-overs. Such cross-overs will, in general, contribute non-perturbatively to the running of $g^2_{\rm SF}(l)$. Although one might argue that such contributions will be negligible at the level of accuracy we are aiming for here, this needs to be checked and, in any case, this raises interesting issues (in its own right) which need to be addressed.

3.5 Other systematic errors

There are a number of other, relatively straightforward, systematic errors that we now address. These include errors from neglecting higher order corrections in $1/l$ in eq. (3.1) when extracting $a^2\sigma$ from the loop masses, and also errors in the actual calculation of these loop masses. There is the error due to the neglect of higher order corrections in a^2 in eq. (3.2). Then there is an error in using only a leading $O(1/N^2)$ correction in eq. (3.3). Finally there are the errors arising from our ignorance of higher order perturbative corrections in eq. (3.2), within the mean-field improved coupling scheme. Assuming this to be a 'good' scheme, as argued above, these last corrections can be plausibly bounded.

We can obtain an estimate of the magnitude of the correction to eq. (3.1) from the calculation in [26]. One finds that for SU(6), and SU(4), the extra correction can be fitted by a term that one can rewrite as $-1.23(21)/a^2\sigma L^3$. For our lattices $a^2\sigma L^2 \sim 10$, so this corresponds to a $\sim 10\%$ addition to the bosonic string correction, and hence a $\sim 0.5\%$ increase in the final estimate of $a\sqrt{\sigma}$. This provides us with an estimate of our systematic error from this source.

We extract the loop masses by identifying effective mass plateaux in appropriate correlators obtained from a variational calculation. In practice the best variational ground state has an overlap of $\sim 99\%$ onto the true ground state so any error should be very small. In the technically very similar case of $D = 2+1$ that was analysed in [29], the shift in the extracted value of $a\sqrt{\sigma}$ induced by excited string states was shown to be less than 0.5%. We can take this as a bound on the corresponding systematic error in the present calculation. Note that this error will decrease the string tension and will therefore partially cancel against the error discussed in the previous paragraph.

The fitted values of $\sqrt{\sigma}/\Lambda_I$ turn out to be very robust against the inclusion of additional $O(a^4)$ corrections in eq. (3.2), with shifts that are typically $\sim 0.25\%$.

The perturbative expression in eq. (3.2) is missing an extra factor $\sim \left(1 + \sum d_n g^{2n}\right)$ that arises from the unknown higher order perturbative corrections. (Ignoring complications that arise from the - at best - asymptotic nature of the expansion.) One might imagine that the simplest way to proceed is to try and fit the first couple of terms in this series. This turns out not to work. The reason is that over our range of scales, the coupling $g^2(a)$ varies very little; thus the correction term is to first approximation just a constant $\left(1 + \sum d_n g^{2n}\right) \sim \left(1 + \sum d_n \overline{g^{2n}}\right)$ which renormalises the fitted value of the overall constant coefficient $\sqrt{\sigma}/\Lambda_I$. For example, if we remove from our fit the 3-loop factor $\exp\{-\beta_2^I g_I^2/2\beta_0^2\}$ in eq. (3.2) we find that we still get a perfectly good fit, but with a value for $\sqrt{\sigma}/\Lambda_I$ that is increased by a factor ~ 1.14. Now if we use, say, the value $g_I^2 N \simeq 5.36$ that we get at $\beta = 44.35$ in table 5, we find that $\exp\{-\beta_2^s g_I^2/2\beta_0^2\} \sim 1.17$ which is numer-

ically similar. What this means is that neglecting higher order perturbative corrections effectively shifts $\sqrt{\sigma}/\Lambda_I$ away from its true value. In the example above, neglecting the 3-loop contribution induces a systematic error in the evaluation of $\sqrt{\sigma}/\Lambda_I$ that is roughly 15%. If we have a well-behaved coupling scheme, one might hope that the error due to the neglect of 4-loop and higher order corrections will be roughly the square of this, i.e. $\sim 2-3\%$. In fact if we estimate, in the above way, the shift induced by the $O(g^4)$ term in eq. (2.3) in the $\overline{MS}$ scheme, where the necessary 4-loop calculation has been performed [20], we find a shift of this magnitude. Assuming that our mean-field improved scheme is also a good one, it then seems safe to bound the systematic error from this source by $\sim \pm 4\%$.

While our 2-loop expression in eq. (3.2) is exact, the 3-loop expression is kept only to leading order in g^2. This makes perfect sense since at higher orders in g^2 one obtains the unknown contributions from higher loops discussed above. Nonetheless we have checked the difference it makes to our fits and extrapolations if we do employ an exact 3-loop expression (by using an expansion to much higher order in g^2) and what we find is that the shift in calculated quantities is only at the $\sim 0.5\%$ level for both the SF and mean-field coupling schemes.

Finally, we find that the inclusion of an additional $O(1/N^4)$ correction in the large-N extrapolation in eq. (3.3) leads to a shift of no more than $\sim 0.5\%$.

Note that all the systematic errors we discuss will to a first approximation be independent of N and should therefore not affect the quality of the large-N extrapolation, but will merely add extra uncertainties to the fitted values.

We thus estimate a reasonable bound on the systematic error from these sources to be $\pm 5\%$. Adding (in quadrature) the $\sim 6\%$ systematic error coming from the choice of scheme (as estimated in section 3.3) we shall take $\pm 8\%$ as our total systematic error.

4. Conclusions

As a by-product of the calculations in section 3.3 we obtained an updated value for the relationship between the two standard scales r_0 and $\sqrt{\sigma}$ in the continuum limit: $r_0\sqrt{\sigma} = 1.160(6)(6)$ where the first error is statistical and the second is systematic.

One of the main purposes of our calculation was to determine the importance of lattice corrections in the relationship between $g^2(a)$ and the lattice spacing a, where $g^2(a)$ is the bare lattice coupling, or some improvement thereof. The novelty of our approach is to do so for SU(6) and SU(8), where the first-order strong-to-weak coupling transition gives unambiguous guidance as to where a weak-coupling fit should be applicable. By contrast, in SU(3) the presence of a smooth strong-to-weak coupling cross-over makes it difficult to evaluate the apparent success or failure of different weak-coupling fits.

We found that, at larger N, the variation of $a\sqrt{\sigma}$ with $g^2(a)$ did indeed demand the presence of an $O(a^2)$ correction to the 3-loop perturbative running, and that its coefficient is $O(1)\times\sigma$ as one would naively expect. Indeed, such a fit accurately describes the running of $g^2(a)$ along the whole weak-coupling branch, including the metastable portion that extends well beyond the location of the bulk transition.

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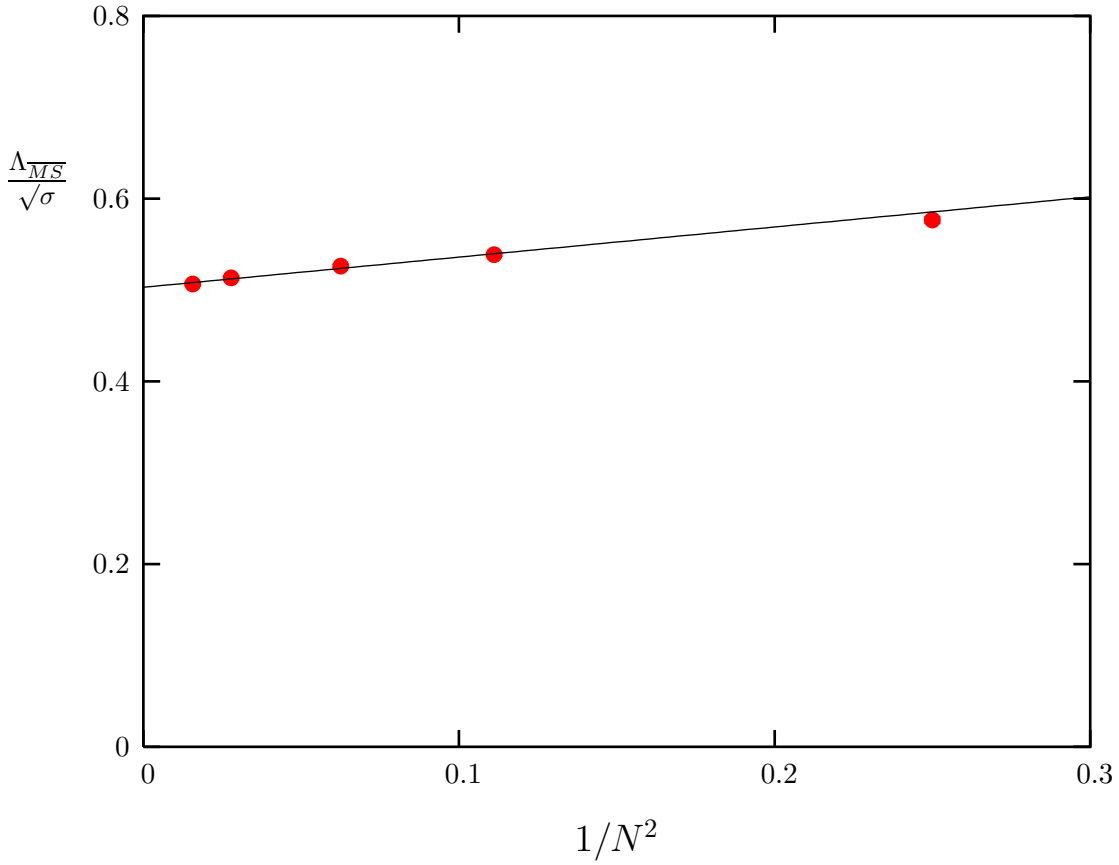

Figure 7: Calculated values of $\Lambda_{\overline{MS}}/\sqrt{\sigma}$ versus $1/N^2$ with a linear extrapolation to $N = \infty$ shown.

As we go to lower N, we find that we need to go deeper into weak-coupling before such weak coupling fits start working. Indeed for SU(2), it is not clear if there is a significant such region below $\beta = 2.70$ (the weakest coupling at which we have a calculation of $a^2\sigma$). An interesting question concerns the functional form in the extended cross-over region where the strong coupling expansion gradually transforms into a weak-coupling one. As we discussed in section 2, a good place to begin such an analysis is in the $D = 1 + 1$ case where analytic arguments are possible [30].

Having a well-motivated lattice correction [9] to the running coupling, invites us to address the more ambitious goal of calculating $\Lambda_{\overline{MS}}/\sqrt{\sigma}$ for various N, so as to interpolate and extrapolate to all N, including $N = \infty$. However in the range of scales $a\sqrt{\sigma}$ where calculations exist, different perturbative schemes give quite different results. This well-known problem is usually addressed by 'improving' the lattice bare coupling. However this turns not to be sufficient. Even if we use variations on the improvement that leave the Λ parameter unchanged, one can easily have a factor of two variation in the fitted $\Lambda_{\overline{MS}}/\sqrt{\sigma}$, as we saw with the otherwise well-motivated mean-field improved coupling. Unfortunately our large-N calculations turn out not to be able to discriminate between these different

coupling schemes — typically they all work almost equally well as fits. To finesse this impasse, we turned to SU(3) and compared a number of variations on the mean-field scheme with the Schrodinger functional (SF) scheme [13] which has been calculated over a range of energy scales comparable to that of experimental determinations of $\alpha_s(Q^2)$ and with greater precision. Applied to calculated values of r_0 this comparison indicated that the original Parisi mean-field scheme works well. Using this scheme for all our values of N, we obtain the fit to all $N \geq 3$ shown in figure 7:

$$\frac{\Lambda_{\overline{MS}}}{\sqrt{\sigma}} = 0.503(2)(40) + \frac{0.33(3)(3)}{N^2} \qquad ; \qquad N \geq 3 \tag{4.1}$$

where the first error is statistical and the second much larger error is expected to provide a bound on the systematic error from all sources. (A fit including the $N = 2$ value is not statistically excluded, but we prefer not to include it given the marginal nature of the SU(2) calculation.) Note that the component systematic errors are independent of N to a first approximation, and we can therefore include that error as a common factor to the best fit in eq. (4.1).

We also presented a way of comparing different coupling schemes that does not depend on the explicit calculation of any physical quantities. This means that one can perform the comparison much deeper into weak coupling. Here again we found evidence that the mean-field improved coupling is a 'good' one in the sense of possessing small higher order corrections.

Our calculations have, of course, have been limited to a small set of improved lattice couplings. There will certainly be other 'good' lattice couplings, and the methods described in this paper can help to identify them. We have also focussed on one particular lattice action (albeit the one that has been most widely used); however it is straightforward to construct a Mean-Field improved coupling for other actions, in the spirit of [7].

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